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The Numerical Solution of Integro-Differential Equations of Parabolic Type

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Abstract

Partial integro-differential equations (PIDEs) arise widely in mathematical models of certain physical phenomena, biological models and chemical kinetics. The main aim of this study is to investigate numerical solution of the partial integro-differential equations of parabolic type, as the PIDE contains two deferent parts, differential part and integral part, we use the fourth order finite difference method to have accurate results for the differential part, and composite weighed trapezoidal rule to calculate the integral part. We solve the integro-differential equations considering the initial-boundary value problems.

Key words: integral-equations, integro-differential equations, finite difference method.

الحل العددي للمعادلات التفاضلية التكاملية من النوع المكافئ

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الملخص

تنشأ المعادلات التفاضلية الجزئية التكاملية (PIDEs)على نطاق واسع في النماذج الرياضية لبعض الظواهر الفيزيائية، والنماذج البيولوجية، والحركية الكيميائية. الهدف الرئيسي لهذه الدراسة هو إيجاد الحل العددي للمعادلات التفاضلية الجزئية التكاملية من النوع المكافئ، حيث تحتوي PIDEs على جزئيين مختلفين، جزء تفاضلي وجزء تكاملي، يتم استخدم طريقة الفروق المنتهية من الدرجة الرابعة للحصول على الحل العددي للجزء التفاضلي، وقاعدة شبه المنحرف الموزونة المركبة لحساب الحل العددي للجزء التكاملي، وتم حل مسألة القيمة الأولية الحدية للمعادلات التفاضلية الجزئية التكاملية.

الكلمات المفتاحية: المعادلات التكاملية، المعادلات التفاضلية الجزئية التكاملية، خوارزمية الفروق المنتهية.

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1. Introduction

Any equation in which the unknown function appears under the sign of integration is called an integral equation (ID). And any integral function associated with a partial differential function with a boundary and initial conditions is called a partial intrgro-differential equation associated by a boundary and initial conditions.

The integro-differentia equation (IDE)is an equation that includes integration and derivation at the same time. IDEs model appear in many situations in science and engineering. Particular integro-differential equations arise widely in mathematical models of certain physical phenomena, biological models and chemical kinetics.

the analytical solutions of some integro-differential equations are difficult to found, thus numerical methods are required.

2. The Previous Studies

Many researchers have been studied the numerical solution of PIDEs by different methods, the parabolic equation with nonlocal boundary conditions has been treated extensively by finite difference methods, finite element procedures, boundary element techniques, spectral schemes, Adomian decomposition method, and the semi discretization procedures in the last 20 years.

(Dehgan et. all 2007) investigated this type of problems and presented finite difference schemes for numerical solution of hyperbolic equations arising boundary value problems with integral condition. The numerical techniques developed by (Dehgan et. all 2007) are based on three-level explicit finite difference procedures.

A different approach is used by using combined finite difference and spectral methods for solving the hyperbolic equation with integral condition.(Pradhan et. all 2008)

The proof of the existence, uniqueness and continuous dependence of the strong solution up on the data for an initial-boundary value problem and integral conditions for this problem is studied by (Dehghan 2006). (Mazumder 2016) uses finite element methods for solving PIDEs of parabolic type. In 2012 (Soliman et. all 2012) introduced the numerical solution of PIDEs of parabolic type, they use the compact finite difference schemes for solving a class of PIDEs.

3. Finite Difference Method (FDM):

The idea of FDM is to replace the partial derivatives of dependent variable (unknown function) with partial differential equation using finite difference approximations with $O(h^n)$ errors (where the independent variables in PDE are defined on) to grid of points where the dependent variables are approximated. The replace of partial derivatives with difference approximation formulas depends on Taylor's theorem as follows

$$\frac{\partial u}{\partial x}(x_i, y_j) = \frac{u(x_{i+1}, y_j) - u(x_i, y_j)}{h} + O(h^2) \qquad (1)$$

this equation is the front differences.

$$\frac{\partial u}{\partial x}(x_i, y_j) = \frac{u(x_i, y_j) - u(x_{i-1}, y_j)}{h} + O(h^2) \qquad (2)$$

this equation is the background differences.

$$\frac{\partial u}{\partial x}(x_i, y_j) = \frac{u(x_{i+1}, y_j) - u(x_{i-1}, y_j)}{2h} + O(h^2) \qquad (3)$$

this equation is the central differences.
We use the method to find the second derivative

$$\frac{\partial^2 u}{\partial x^2}(x_i, y_j) = \frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{2h^2} + O(h^2) \quad (4)$$

by using the Taylor’s series expansion, a fourth orders accurate finite difference for the first and second derivatives can be approximated by

$$\frac{\partial u}{\partial x} = \frac{du}{dx} + \frac{h^2}{3!}\frac{d^3u}{dx^3} = \left(1 + \frac{h^2}{6}\frac{d^2}{dx^2}\right)\frac{d}{dx}$$

$$= \left(1 + \frac{h^2}{6}\partial^2\right)\frac{d}{dx} + O(h^4) \qquad (5)$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2u}{dx^2} + \frac{h^2}{12}\frac{d^4u}{dx^4} = \left(1 + \frac{h^2}{12}\frac{d^2}{dx^2}\right)\frac{du^2}{dx^2}$$

$$= \left(1 + \frac{h^2}{12}\partial^2\right)\partial^2 u + O(h^4) \qquad (6)$$

Definition 1

The shape functions (φi) are derived from a set of kernel functions, also known as weight functions, denoted $\varphi i\colon \Omega \to R$. Finally, each weight function has a certain shape, required to be continuous and positive. The continuity of the shape functions only depends on the continuity of the kernel functions.

A linear integro-differential equation is an equation of the form

$$a_0(x)u^{(n)}(x) + a_1(x)u^{(n-1)}(x) + \cdots + a_n(x)u(x) =$$

$$\sum_{m=0}^{s}\int_0^x k_m(x,t)\,u^{(m)}(t)dt + f(x) \qquad (7)$$

Where $a_0(x), a_1(x), \ldots a_n(x)$ are known function or constant coefficients. $k_m(x,t)$known function and $u(x)$ is the unknown function.

$$u(0) = u_0, u'(0) = u_0' , \dots u^{(n-1)}(0) = u_0^{(n-1)}$$

are the initial conditions.
This type of integrative equation can be type of Friedholm equations or Volterra equations.

4. Integro-Differential Equations:

The general form of the Integro- Differential equations is given by

$$u^{(n)}(x) = \int_0^x k_m(x,t)\, u^{(m)}(t) dt + f(x) \qquad (8)$$

We will study the parabolic kind of PIDEs that have the form;

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\partial u}{\partial x} = \int_0^t k(t,s) u(x,s) ds\,, + f(x,t), x \in [a,b],$$
$$t \in (0,T), \qquad (9)$$

which is associated by the following initial condition:

$$u(x,0) = u_0(x), \qquad (10)$$

and the boundary conditions:

$$u(a,t) = 0, u(b,t) = 0 \qquad (11)$$

where $a \leq x \leq b$, $0 \leq t < T$ and $0 < s < T$. $u(x,t)$and $f(x,t)$are continuous function. t is a time , $k(t,s)$ is the kernel of PPIDE. $f(x,t)$ is the non-homogeneous term, the integral term is called memory term.

5. Compact finite difference method for solving parabolic PIDE

In this section we use the fourth order compact finite difference method to solve problem given in Eq.(9) and Eq.(10), which is finding $u(x,t)$,
To construct a numerical solution, we consider the $(x_j, t_i) \in [a,b] \times [0,T]$ which is called the nodal points, where τ is the time step given by,

$$\tau = t_{i+1} - t_i, i = 0,1,2, \dots$$

and h is the spatial discretization step,

$$h = \Delta x = x_{j+1} - x_j, \quad j = 0,1,2,\dots,n$$

h and τ are strictly positive real numbers.
so, we have in such case;

$$x_j = a + jh, \quad j = 0,1,2,\dots n \qquad t_i = i\tau, \qquad i = 0,1,2,\dots$$

from that, the initial condition in Eq.(10) becomes;

$$u(x,0) = u_0 = u(x,t_0). \tag{12}$$

Some important notations which we will use them;

$$u_j = u(x_j), \tag{13}$$

i) $\frac{\partial u}{\partial x}(x_j,t_i) \approx \frac{u_{j+1}-u_j}{h} = \delta_{x+}u_j$, (14)

the standard forward finite difference scheme.

ii) $\frac{\partial u}{\partial x}(x_j,t_i) \approx \frac{u_j-u_{j-1}}{h} = \delta_{x-}u_j$, (15)

the standard backward finite difference scheme.

iii) $\frac{\partial u}{\partial x}(x_j,t_i) \approx \frac{u_{j+1}-u_{j-1}}{2h} = \frac{1}{2}(\delta_{x+}u_j - \delta_{x-}u_j) = \delta_0 u_j$, (16)

the first-order centered finite difference with respect to x.

iv) $\frac{\partial^2 u}{\partial x^2}(x_j,t_i) \approx \frac{u_{j+1}-2u_j+u_{j-1}}{h^2} = \frac{\delta_{x+}u_j-\delta_{x-}u_j}{h^2} = \delta_{x+}\delta_{x-}u_j$, (17)

the second-order centered finite difference with respect to x.

Now, we need to do the same for the partial derivatives with respect to t:

v) $\frac{\partial u}{\partial t}(x_j,t_i) \approx \frac{u_{i+1}-u_i}{\tau} = \delta_{t+}u_i$, (18)

the standard forward finite difference in t.

vi) $\delta_x u = \frac{du}{dx} + \frac{h^2}{2!}\frac{d^2u}{dx^2} = \left(1 + \frac{h^2}{6}\frac{d^3u}{dx^3}\right)\frac{du}{dx} = \left(1 + \frac{h^2}{6}\delta^2\right)\frac{du}{dx} + o(h^4)$, (19)

a fourth orders accurate finite difference for the first derivative using the Taylor's series expansion.

vii) $\delta_x^2 u = \frac{d^2u}{dx^2} + \frac{h^2}{12}\frac{d^4u}{dx^4} = \left(1 + \frac{h^2}{12}\frac{d^2}{dx^2}\right)\frac{d^2u}{dx^2} = \left(1 + \frac{h^2}{12}\delta^2\right)\delta^2 u + o(h^4)$, (20)

a fourth orders accurate finite difference for the second derivative.

Now, we approximate $\frac{\partial u}{\partial t} = u_t$ in Eq.(9) at the time-level $t_i, i = 0,1,2,\dots$ using Eq.(18), this means that Eq. (9) is discretized in by time replace the partial derivative u_t in Eq.(9) by Eq.(19) as follow

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$$\frac{u_{i+1}(x)-u_i(x)}{\tau}-\frac{d^2u_{i+1}(x)}{dx^2}+\alpha\frac{du_i(x)}{dx}$$
$$=\int_0^{t_{i+1}} k_{i+1}(s)u(x,s)ds+f_{i+1}(x), \qquad (21)$$

where

$$k_{i+1}(s)=k(t_{i+1},s), f_{i+1}(x)=f(x,t_{i+1}),\ u_{i+1}(x)=u(x,t_{i+1}), \qquad (22)$$

Eq. (21) is simplified to

$$u_{i+1}-\frac{d^2u_{i+1}}{dx^2}+u_i^*(x)$$
$$=\tau\int_0^{t_{i+1}} k_{i+1}(s)u(x_i,s)ds+\tau f_{i+1}(x), \qquad (23)$$

with

$$u_i^*(x)=\tilde{u}_i(x-\tau\alpha),$$

where$\tilde{u}_i$ an extension of u_i for some details of $\tilde{u}$.
We can rewrite the Eq. (23) as

$$u''_{i+1}(x)=\frac{u_{i+1}(x)-u_i^*(x)}{\tau}-\int_0^{t_{i+1}} k_{i+1}(s)u(x,s)ds$$
$$-f_{i+1}(x), \qquad (24)$$

we put $x=x_j, j=1,\dots,n-1$ in the initial condition we get;

$$u(x_j,t_0)=u_{t_0}(x_j), \qquad (25)$$

so we have from equations (14),(22) and (25) that

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$$u_{i+1}(x) = u_{i+1}(x_j) = u(x_j, t_{i+1}) = u_{i+1,j}, \qquad (26)$$

$$u_i^*(x) = u_i^*(x_j) = u^*(x_j, t_i) = u_{i,j}^*, \qquad (27)$$

$$f_{i+1}(x) = f_{i+1}(x_j) = f(x_j, t_{i+1}) = f_{i+1,j}, \qquad (28)$$

$$u(x,s) = u(x_i, s), \qquad (29)$$

from the last notations, Eq. (24) becomes as follows:

$$u''_{i+1,j} = \frac{u_{i+1,j} - u_{i,j}^*}{\tau} - \int_0^{t_{i+1}} k_{i+1}(s) u(x_i, s) ds - f_{i+1,j}, j = 0, \dots, n, \quad (30)$$

using Eq.(26) to replace $\delta_x^2 u$ in Eq. (20) by $\delta_x^2 u_{i+1,j}$, so the Eq. (20) becomes

$$\delta_x^2 u_{i+1,j} = u''_{i+1,j} + \frac{h^2}{12}\delta_x^2 u''_{i+1,j},$$

$$u''_{i+1,j} = \frac{\delta_x^2 u_{i+1,j}}{1 + \frac{h^2}{12}\delta_x^2}, \qquad (31)$$

Substituting $u''_{i+1,j}$ from Eq. (31)in to Eq.(30) we get

$$\frac{\delta_x^2 u_{i+1,j}}{1+\frac{h^2}{12}\delta_x^2} = \frac{u_{i+1,j} - u_{i,j}^*}{\tau} - \int_0^{t_{i+1}} k_{i+1}(s) u(x_j, s) ds - f_{i+1,j}, \qquad (32)$$

now, multiplying the last Eq. (32) by $\left(1 + \frac{h^2}{12}\delta_x^2\right)$, we get

$$\delta_x^2 u_{i+1,j} = \left(1 + \frac{h^2}{12}\delta_x^2\right)\frac{u_{i+1,j}}{\tau} - \left(1 + \frac{h^2}{12}\delta_x^2\right)\frac{u_{i,j}^*}{\tau} -$$

$$\left(1+\frac{h^2}{12}\delta_x^2\right)\int_0^{t_{i+1}} k_{i+1}(s)u(x_i,s)ds-\left(1+\frac{h^2}{12}\delta_x^2\right)f_{i+1,j},$$

For more simply

$$\Rightarrow\left(1-\frac{h^2}{12\tau}\right)\delta_x^2 u_{i+1,j}-\frac{u_{i+1,j}}{\tau}$$
$$+\int_0^{t_{i+1}} k_{i+1}(s)u(x_j,s)ds+\frac{h^2}{12}\int_0^{t_{i+1}} k_{i+1}(s)\delta_x^2 u(x_j,s)ds=$$
$$=-\frac{u_{i,j}^*}{\tau}-\frac{h^2}{12\tau}\delta_x^2 u_{i,j}^*-f_{i+1,j}$$
$$-\frac{h^2}{12}\delta_x^2 f_{i+1,j}, \qquad (33)$$

we have from Eq.(17) that

$$\delta_x^2 u_j=\frac{u_{j+1}-2u_j+u_{j-1}}{h^2},$$

so

$$\delta_x^2 u_{i+1,j}=\frac{u_{i+1j+1}-2u_{i+1,j}+u_{i+1,j-1}}{h^2}, \qquad (34)$$

$$\delta_x^2 u(x_i,s)=\frac{u_{j+1}-2u_j+u_{j-1}}{h^2}, \qquad (35)$$

and

$$\delta_x^2 u_{i,j}^*=\frac{u_{i,j+1}^*-2u_{i,j}^*+u_{i,j-1}^*}{h^2}, \qquad (36)$$

substituting Eq.(34),(35), and Eq. (36) in Eq.(33) we get;

$$\left(\frac{1}{h^2}-\frac{1}{12\tau}\right)\left(u_{i+1,j+1}+u_{i+1,j-1}\right)+\left(\frac{-2}{h^2}-\frac{5}{6\tau}\right)u_{i+1,j}$$
$$+\frac{5}{6}\int_0^{t_{i+1}} k_{i+1}(s)u_j(s)ds \; + + \frac{1}{12}\int_0^{t_{i+1}} k_{i+1}(s)u_{j+1}(s)ds$$
$$+\frac{1}{12}\int_0^{t_{i+1}} k_{i+1}(s)u_{j-1}(s)ds =$$
$$=\frac{-1}{12\tau}\left(u^*_{i,j+1}+u^*_{i,j-1}\right)-\frac{5}{6\tau}u^*_{i,j}-\frac{1}{12}\left(f_{i+1,j+1}+f_{i+1,j-1}\right)$$
$$-\frac{5}{6}f_{i+1,j},) \quad (37)$$

6. The composite weighted trapezoidal base:

we will use the composite weighted trapezoidal rule for calculating the integral term. This rule given by

$$\int_{t_0}^{t_{i+1}} f(s)ds \approx \tau\sum_{m=0}^{i}[wf(t_m)+(1-w)f(t_{m+1})]=\tau\big[wf(t_0)+ (1-w)f(t_{i+1})++\sum_{m=1}^{i} f(t_m)\big] \quad (38)$$

using Eq.(38) and the initial condition Eq.(10) for given integral as follow

$$\int_0^{t_{i+1}} k_{i+1}(s)u(x,s)ds$$
$$\approx \tau\left[wk_{i+1}(0)u_0(x)+(1-w)k_{i+1}(t_{i+1})u_{i+1}(x) + \sum_{m=1}^{i} k_{i+1}(t_m)u_{i+1-m}(x)\right], \quad (39)$$

substituting Eq.(39) into Eq.(37) we get

$$\left(\frac{1}{h^2}-\frac{1}{12\tau}\right)\left(u_{i+1,j+1}+u_{i+1,j-1}\right)+\left(\frac{-2}{h^2}-\frac{5}{6\tau}\right)u_{i+1,j}$$
$$+\frac{5\tau}{6}\left[wk_{i+1}(0)u_{0,j}+(1-w)k_{i+1}(t_{i+1})u_{i+1,j}\right.$$
$$\left.+\sum_{m=1}^{i}k_{i+1}(t_m)u_{i+1-m,j}\right]$$
$$+\frac{\tau}{12}\left[wk_{i+1}(0)u_{0,j+1}+(1-w)k_{i+1}(t_{i+1})u_{i+1,j+1}\right.$$
$$\left.+\sum_{m=1}^{i}k_{i+1}(t_m)u_{i+1-m,j+1}\right]$$
$$+\frac{\tau}{12}\left[wk_{i+1}(0)u_{0,j-1}+(1-w)k_{i+1}(t_{i+1})u_{i+1,j-1}\right.$$
$$\left.+\sum_{m=1}^{i}k_{i+1}(t_m)u_{i+1-m,j-1}\right]=\frac{-1}{12\tau}\left(u^*_{i,j+1}+u^*_{i,j-1}\right)-$$
$$-\frac{5}{6\tau}u^*_{i,j}-\frac{1}{12}\left(f_{i+1,j+1}+f_{i+1,j-1}\right)-\frac{5}{6}f_{i+1,j},\quad(40)$$

then, we will use the collocation method to obtain system of algebraic equations, then we will find the unknown function.

7. The approximates function:

Let $U_i(x)$ be a function which approximates $u(x_j,t_i)$ for the time-level $t_i=i\tau$, $\left(U_i(x)\approx u(x,t_i)\right)$. It is a linear combination of $n+1$ shape functions which is expressed as:

$$U_i(x) = \sum_{m=0}^{n} C_{mi}\psi_m(x), \tag{41}$$

where$(C_{mi})_{m=0}^{n}$ are the unknown real functions which we want to compute it, and $\psi_m(x)$ are any known basis functions.

First we approximate the solution U_{i+1} for $i = 0$ in Eq. (40), by U_1 as is given in Eq. (41), then Eq. (40) is approximated by

$$\left(\frac{1}{h^2} - \frac{1}{12\tau}\right)\left(U_{1,j+1} + U_{1,j-1}\right) + \left(\frac{-2}{h^2} - \frac{5}{6\tau}\right)U_{1,j} + \frac{5\tau}{6}\left[wk_1(0)u_{0,j} + (1-w)k_1(t_1)U_{1,j}\right] + \frac{\tau}{12}\left[wk_1(0)u_{0,j+1} + (1-w)k_1(t_1)U_{1,j+1}\right] + \frac{\tau}{12}\left[wk_1(0)u_{0,j-1} + (1-w)k_1(t_1)U_{1,j-1}\right] = \frac{1}{12\tau}\left(u^*_{0,j+1} + u^*_{0,j-1}\right) - \frac{5}{6\tau}u^*_{0,j} - \frac{1}{12}\left(f_{1,j+1} + f_{1,j-1}\right) - \frac{5}{6}f_{1,j}, \tag{42}$$

now, we replace U_1 by the approximate solution given by eq. (31), that is;

$$U_1(x) = \sum_{m=0}^{n} C_{m1}\psi_m(x), \tag{43}$$

we get the follow linear system of $n-1$ equations:

$$\left(\frac{1}{h^2} - \frac{1}{12\tau}\right)\left(\sum_{m=0}^{n} C_{m1}\psi_{m,j+1} + \sum_{m=0}^{n} C_{m1}\psi_{m,j-1}\right) + \left(\frac{-2}{h^2} - \frac{5}{6\tau}\right)\left(\sum_{m=0}^{n} C_{m1}\psi_{m,j}\right) + \frac{5\tau}{6}\left[wk_1(0)u_{0,j} + (1-w)k_1(t_1)\sum_{m=0}^{n} C_{m1}\psi_{m,j}\right]$$

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$$+\frac{\tau}{12}\left[wk_1(0)u_{0,j+1}+(1-w)k_1(t_1)\sum_{m=0}^{n}C_{m1}\psi_{m,j+1}\right]$$
$$-\frac{1}{12}\tau\left(u^*_{0,j+1}+u^*_{0,j-1}\right)-\frac{5}{6\tau}u^*_{0,j}-\frac{1}{12}\left(f_{1,j+1}+f_{1,j-1}\right)$$
$$-\frac{5}{6}f_{1,j},$$

$$\Rightarrow\left(\frac{1}{h^2}-\frac{1}{12\tau}\right)\left(\sum_{m=0}^{n}C_{m1}\psi_{m,j+1}+\sum_{m=0}^{n}C_{m1}\psi_{m,j-1}\right)$$
$$+\left(\frac{-2}{h^2}-\frac{5}{6\tau}\right)\left(\sum_{m=0}^{n}C_{m1}\psi_{m,j}\right)$$
$$+\frac{5\tau}{6}\left[(1-w)k_1(t_1)\sum_{m=0}^{n}C_{m1}\psi_{m,j}\right]$$
$$+\frac{\tau}{12}\left[(1-w)k_1(t_1)\sum_{m=0}^{n}C_{m1}\psi_{m,j+1}\right]$$
$$+\frac{\tau}{12}\left[(1-w)k_1(t_1)\sum_{m=0}^{n}C_{m1}\psi_{m,j-1}\right]$$
$$=\left(\frac{-5\tau wk_1(0)}{6}\right)u_{0,j}-\frac{5}{6\tau}u^*_{0,j}$$
$$-\left(\frac{\tau wk_1(0)}{12}\right)\left(u_{0,j+1}+u_{0,j-1}\right)-\frac{1}{12\tau}\left(u^*_{0,j+1}+u^*_{0,j-1}\right)$$
$$-\frac{1}{12}\left(f_{1,j+1}+f_{1,j-1}\right)-\frac{5}{6}f_{1,j},\qquad(44)$$

note that;

$$\sum_{m=0}^{n}C_{m1}\psi_{m,j+1}=\sum_{m=0}^{n}C_{m1}\psi_m(x_{j+1}),$$

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we can rewrite Eq.(44) as follows:

$$\sum_{m=0}^{n} C_{m1}\left(a_1\psi_{m,j+1} + a_2\psi_{m,j} + a_1\psi_{m,j-1}\right)$$
$$= a_2u_{0,j} + a_4\left(u_{0,j+1} + u_{0,j-1}\right) + a_5u^*_{0,j}$$
$$+ a_6\left(u^*_{0,j+1} + u^*_{0,j-1}\right)$$

$$+a_6\tau\left(f_{1,j+1} + f_{1,j-1}\right) + a_5\tau f_{1,j}, \qquad (45)$$

for $i_0 = 0$, where;

$$\left.\begin{aligned} a_1 &= \frac{1}{h^2} - \frac{1}{12\tau} + \frac{\tau}{12}(1-w)k_1(t_1) \\ a_2 &= \frac{-2}{h^2} - \frac{5}{6\tau} + \frac{5\tau}{6}(1-w)k_1(t_1) \\ a_2 &= \frac{-5\tau w k_1(0)}{6} \\ a_4 &= \frac{-\tau w k_1(0)}{12} \\ a_5 &= \frac{-5}{6\tau} \\ a_6 &= \frac{-1}{12\tau} \end{aligned}\right\}, \qquad (46)$$

note that, Eq. (45) is system consists of $(n-1)$ of equations in the $(n+1)$ of unknowns $(C_{m1})_{m=0}^{n}$. So we must find a solution of this system. To do that we need two additional conditions. We get these conditions from the boundary conditions (10) as follows:

$$u(a, t_i) = \sum_{m=0}^{n} C_{m1}\psi_m(a) = g_1(t_i), i = 0, \dots, n \qquad (47)$$

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$$u(b,t_i)=\sum_{m=0}^{n} C_{m1}\psi_m(b)=g_2(t_i), i=0,\dots,n \qquad (48)$$

since$f$ and u_0 in the right hand side of Eq. (45) are known, so the right hand side of equation (45) is known for all nodes. The system (45) with the boundary conditions (47) and (48) consists of $n-1$ of equations in $n+1$ of known. This system takes the form:

$$AC=F, \qquad (49)$$

Where

$$A=[a_1\psi_{0,j+1}+a_2\psi_{0,j}+a_1\psi_{0,j-1}+a_1\psi_{1,j+1}+a_2\psi_{1,j}+a_1\psi_{1,j-1}+\cdots \\ \dots\dots+a_1\psi_{n,j+1}+a_2\psi_{n,j}+a_1\psi_{n,j-1}]$$

$$C_{m1}=\begin{bmatrix} C_{01} \\ C_{11} \\ \vdots \\ C_{n1} \end{bmatrix}$$

$$F=\left[a_2u_{0,j}+a_4\left(u_{0,j+1}+u_{0,j-1}\right)+a_5u^*_{0,j}+a_6\left(u^*_{0,j+1}+u^*_{0,j-1}\right) \\ +a_6\tau\left(f_{1,j+1}+f_{1,j-1}\right) \\ +a_5\tau f_{1,j}\right]$$

Note that

$$U_1(x)=U_1(x_j)=\sum_{m=0}^{n} C_{mi}\psi_m(x_j), =0,\dots,n, \qquad (50)$$

we found the approximate solution at time-level t_0. Now, we will find the approximate solution at time-level $t_1, t_2, \dots$ recursively by solve the follow system for $i=1,2,\dots$ (this system results from Eq. (31) by replacing

$$u_{i+1,j+1}, u_{i+1,j}, u_{i+1,j-1}, \dots$$

by

$$U_{i+1,j+1}, U_{i+1,j}, U_{i+1,j-1}, \dots$$

Problem 1: We consider

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\partial u}{\partial x} = \int_0^t k(t,s)u(x,s)ds + f(x,t), 0 \le x \le 1 , 0 \le t < T \ (1)$$

which is associated by the following initial condition:

$$u(x,0) = u_0(x),$$

and the boundary conditions $u(a,t) = 0 , u(b,t) = 0$

$$k(t,s) = e^{-\pi^2(t-s)}$$

$$f(x,t) = \alpha\pi e^{-\pi^2(t)} \cos(\pi x) - t e^{-\pi^2(t)} sin(\pi x)$$

$$u(0,t) = 0 , \qquad u(1,t) = 0$$

$$u(x,0) = sin(\pi x) , \quad 0 \le x \le 1. \quad g_1(t) = g_1(t) = 0.$$

Exact solution is $u(t,x) = e^{-\pi^2(t)} sin(\pi x)$

Solution

Numerical results of Eq.(1) using Matlab programming shown in the following table, also shown the comparison between the approximate solution and the exact solution

Value	U_approximate	U_exact	Error
0	1.6074e-10-	0	-1.6074e-10
0.1	0.26129	0.25366	0.001103
0.2	0.49493	0.48249	0.0020891
0.3	0.67767	0.6641	0.0027074
0.4	0.78325	0.78069	0.0025551
0.5	0.82814	0.82087	0068687.00
0.6	0.78307	0.78069	-0.0011831
0.7	0.66252	0.6641	-0.0016555
0.8	0.47848	0.48249	-0.0014413
0.9	0.25002	0.25366	-0.00082995
1	-4.3656e-10	e-161.0053	-4.3656e-10

Table (1)Numerical solution of PIDEs when

$$t = 0.02\,, \quad \alpha = 1, \quad \tau = 0.01$$

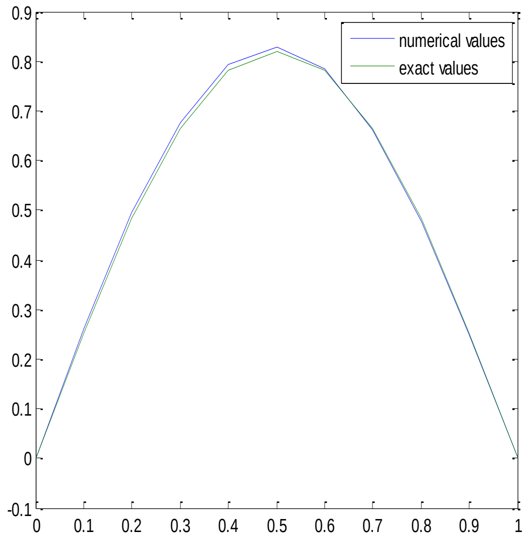

Figure (1)Numerical solution of PIDEs when

$$t = 0.02\,, \quad \alpha = 1, \quad \tau = 0.01$$

Solution problem (1) with different t, α, τ

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Value	U_approximate	U_exact	Error
0	3.8263e-10	0	3.8263e-10
0.1	0.25466	0.25366	0.001103
0.2	0.48458	0.48249	0.0020891
0.3	0.6668	0.6641	0.0027074
0.4	0.78325	0.78069	0.0025551
0.5	0.82156	0.82087	0068687.00
0.6	0.77951	0.78069	-0.0011831
0.7	0.66244	0.6641	-0.0016555
0.8	0.48105	0.48249	-0.0014413
0.9	0.25283	0.25366	-0.00082995
1	e-102.6193	e-161.0053	e-102.6193

Table (2) Numerical solution of PIDEs when

$$t = 0.02,\ \alpha = 1,\ \tau = 0.001$$

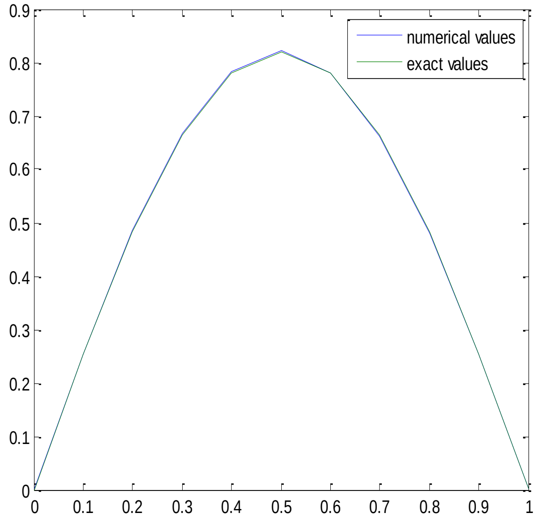

Figure (2)Numerical solution of PIDEs when

$$t = 0.02\ ,\quad \alpha = 1,\quad \tau = 0.001$$

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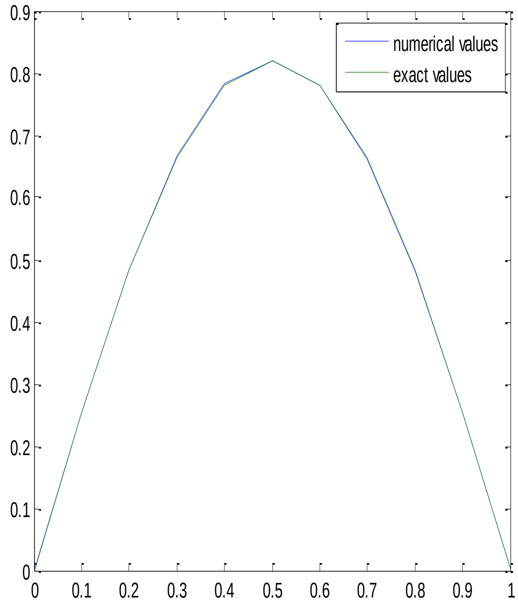
Value	U_approximate	U_exact	Error
0	2.9982e-10	0	2.9982e-10
0.1	0.25436	0.25366	0.00069883
0.2	0.48382	0.48249	0.0013253
0.3	0.66589	0.6641	0.00179
0.4	0.78239	0.78069	0.0016932
0.5	0.82088	0.82087	e-068.291
0.6	0.77906	0.78069	-0.0016326
0.7	0.66224	0.6641	-0.0018574
0.8	0.48103	0.48249	-0.0014668
0.9	0.25284	0.25366	-0.00081808
1	e-102.9104	e-161.0053	e-102.9104

Table(3)Numerical solution of PIDEs when

$$\boldsymbol{t = 0.02\,, \quad \alpha = 1, \quad \tau = 0.0001}$$



Figure(3)Numerical solution of PIDEs when

$$\boldsymbol{t = 0.02\,, \quad \alpha = 1, \quad \tau = 0.0001}$$

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Value	U_approximate	U_exact	Error
0	e-113.1918	0	3.1918e-11
0.1	0.28043	0.27997	0.00045522
0.2	0.53339	0.53254	0.00084889
0.3	0.7342	0.73298	0.0012174
0.4	0.86299	0.86167	0.0013141
0.5	0.90605	0.90602	e-052.8255
0.6	0.86045	0.86167	-0.0012232
0.7	0.73178	0.73298	-0.0012064
0.8	0.53167	0.53254	-0.00087635
0.9	0.27949	0.27997	-0.00048675
1	e-104.0745	e-161.1096	e-104.0745

Table (4)Numerical solution of PIDEs when

$$t = 0.01\,, \quad \alpha = 1, \quad \tau = 0.0001$$

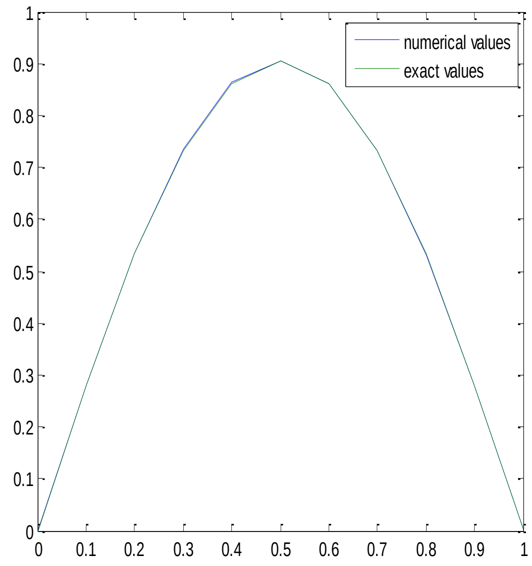

Figure (4)Numerical solution of PIDEs when

$$t = 0.01\,, \quad \alpha = 1, \quad \tau = 0.0001$$

Conclusions

As the analytical solution of the PIDEs takes long time to get, and some of these equations do not have analytical solution, we studied the numerical solution of the problem of PIDE of parabolic type. The fourth order finite difference method was used to obtain accurate results for the differential part, and composite weighed trapezoidal rule to calculate the integral part. we used Matlap to obtain the final numerical solution. The numerical results show that, the approximate solution of the PIDEs of parabolic type is almost the same as the actual solution. As can be seen in figures 1,2,3,4 The more accurate results obtained when the time-step τ is reduced as reported in figure1,2,3,4. So the results in tables 1,2,3,4 confirm that the numerical solutions can be refined when the time-step τ is reduced.

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